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On the heavenly equation hierarchy and its reductions

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Abstract

Second heavenly equation hierarchy is considered using the framework of hyper-Kähler hierarchy developed by Takasaki (1989 *J. Math. Phys.* **30** 1515–21, 1989 *J. Math. Phys.* **31** 1877–88). Generating equations for the hierarchy are introduced, they are used to construct generating equations for reduced hierarchies. General N-reductions, logarithmic reduction and rational reduction for one of the Lax–Sato functions are discussed. It is demonstrated that rational reduction is equivalent to the symmetry constraint.

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1. Introduction

Plebansky second heavenly equation [3], having its origin in general relativity, has attracted a lot of interest both from the viewpoint of integrability and relativity. It has been intensively studied using different techniques (see, e.g., [4–10]).

In the work [11] we have developed a $\bar{\partial}$-dressing scheme applicable to second heavenly equation. A very important role was played by a kind of Hirota bilinear identity, which leads to the introduction of the function Θ (analogue of the τ-function for heavenly equation hierarchy) and produces the hierarchy in the form of addition formulae (generating equations) for Θ. This identity also has its natural place in the framework of hyper-Kähler hierarchy developed by Takasaki [1, 2], who demonstrated that it is equivalent to the Lax–Sato equations of the hierarchy. Here, we will use this framework to study the reductions of the heavenly equation hierarchy and its symmetry constraints. The ideas and logic of this work are very close to the works [12–14], where dispersionless hierarchies were considered.

2. Heavenly equation hierarchy

First we introduce the principal objects and notation. We start from two formal Laurent series in z,

$$S^1 = \sum_{n=0}^{\infty} t_n^1 z^n + \sum_{n=1}^{\infty} S_n^1(\mathbf{t}^1, \mathbf{t}^2) z^{-n}, \tag{1}$$

$$S^2 = \sum_{n=0}^{\infty} t_n^2 z^n + \sum_{n=1}^{\infty} S_n^2(\mathbf{t}^1, \mathbf{t}^2) z^{-n}, \tag{2}$$

where the variables t are considered independent and S_n^1, S_n^2 are the dependent variables. We denote $x = t_0^1$, $y = t_0^2$, $\mathbf{S} = \binom{S^1}{S^2}$, introduce the Poisson bracket $\{f, g\} := f_x g_y - f_y g_x$ and the projectors $\left(\sum_{-\infty}^{\infty} u_n z^n\right)_+ = \sum_{n=0}^{\infty} u_n z^n$, $\left(\sum_{-\infty}^{\infty} u_n z^n\right)_- = \sum_{-\infty}^{n=-1} u_n z^n$.

Heavenly equation hierarchy is defined by the relation (see [1, 11])

$$(\mathrm{d}S^1 \wedge \mathrm{d}S^2)_- = 0, \tag{3}$$

playing a role similar to the role of the famous Hirota bilinear identity for KP hierarchy. This relation is equivalent to the Lax–Sato form of the hierarchy.

Proposition 1. *The identity (3) is equivalent to the set of equations*

$$\partial_n^1 \mathbf{S} = -\{(z^n S^2)_+, \mathbf{S}\}, \tag{4}$$

$$\partial_n^2 \mathbf{S} = \{(z^n S^1)_+, \mathbf{S}\}, \tag{5}$$

$$\{S^1, S^2\} = 1. \tag{6}$$

The proof of this statement is given in [1] for the general hyper-Kähler hierarchy (second heavenly equation hierarchy is its two-component special case). We will not reproduce the complete proof, but will just illustrate some ideas, which will be useful later. It is possible to prove that (3) implies Lax–Sato equations using the following statement.

Lemma 1. *Given identity (3), for arbitrary first-order operator* $\hat{U}$*,*

$$\hat{U}\mathbf{S} = \sum_i \left(u_i^1(z, \mathbf{t}^1, \mathbf{t}^2)\partial_i^1 + u_i^2(z, \mathbf{t}^1, \mathbf{t}^2)\partial_i^2\right)\mathbf{S} \tag{7}$$

with 'plus' coefficients $\left(\left(u_i^1\right)_- = \left(u_i^2\right)_- = 0\right)$*, the equality* $(\hat{U}\mathbf{S})_+ = \mathbf{0}$ *implies that* $\hat{U}\mathbf{S} = \mathbf{0}$*.*

Proof. Identity (3) implies that

$$\left(S_{\tau_1}^1 S_{\tau_2}^2 - S_{\tau_2}^1 S_{\tau_1}^2\right)_- = 0, \tag{8}$$

where τ_1, τ_2 are the arbitrary times of the hierarchy. In particular,

$$\{S^1, S^2\}_- = 0,$$

and, using (1), (2), we get

$$\{S^1, S^2\} = \{S^1, S^2\}_+ = 1. \tag{9}$$

Then, for arbitrary first-order operator $\hat{U}$ of the form (7) with 'plus' coefficients $\left(\left(u_i^1\right)_- = \left(u_i^2\right)_- = 0\right)$ (8) gives

$$\left(\hat{U}S^1 \cdot S_x^2 - \hat{U}S^2 \cdot S_x^1\right)_- = 0, \qquad \left(\hat{U}S^1 \cdot S_y^2 - \hat{U}S^2 \cdot S_y^1\right)_- = 0. \tag{10}$$

Let us suggest that $(\hat{U}\mathbf{S})_+ = \mathbf{0}$. By direct calculation using (1), (2) we check that

$$\left(\hat{U}S^1 \cdot S_x^2 - \hat{U}S^2 \cdot S_x^1\right)_+ = 0, \qquad \left(\hat{U}S^1 \cdot S_y^2 - \hat{U}S^2 \cdot S_y^1\right)_+ = 0,$$

thus (taking into account (10))

$$\hat{U}S^1 \cdot S_x^2 - \hat{U}S^2 \cdot S_x^1 = 0, \qquad \hat{U}S^1 \cdot S_y^2 - \hat{U}S^2 \cdot S_y^1 = 0.$$

If $\hat{U}\mathbf{S} \neq \mathbf{0}$, then the determinant for the system of linear equations should be equal to zero, $\{S^1, S^2\} = 0$, and we come to a contradiction with (9). So $\hat{U}\mathbf{S} = \mathbf{0}$ □

The proof of proposition 1 (sufficient condition) is then straightforward, one should just check directly that

$$\left(\partial_n^1\mathbf{S} + \{(z^n S^2)_+, \mathbf{S}\}\right)_+ = 0, \qquad \left(\partial_n^2\mathbf{S} - \{(z^n S^1)_+, \mathbf{S}\}\right)_+ = 0.$$

2.1. Function Θ

Identity (3) also leads to the introduction of an analogue of the τ-function in terms of closed 1-form.

Proposition 2. *The 1-form*

$$\theta = \frac{1}{2\pi \mathrm{i}}\operatorname{Res}_\infty\left(S_-^2\,\mathrm{d}S_+^1 - S_-^1\,\mathrm{d}S_+^2\right) \tag{11}$$

is closed.

Proof. Identity (3) implies that

$$\mathrm{d}\theta = \frac{1}{2\pi \mathrm{i}}\operatorname{Res}_\infty\left(\mathrm{d}S_-^2 \wedge \mathrm{d}S_+^1 - \mathrm{d}S_-^1 \wedge \mathrm{d}S_+^2\right) = 0.$$

□

Similar to the work [1], we define a τ-function $\Theta(\mathbf{t}^1,\mathbf{t}^2)$ for heavenly equation hierarchy through closed 1-form (11) by the relation $\mathrm{d}\Theta = \theta$. Introducing vertex operators $D^1(z) = \sum_{n=0}^{\infty} z^{-n-1}\partial_n^1$, $D^2(z) = \sum_{n=0}^{\infty} z^{-n-1}\partial_n^2$, it is easy to demonstrate that

$$S_-^1(z) = -D^2(z)\Theta, \qquad S_-^2(z) = D^1(z)\Theta. \tag{12}$$

Substituting this representation into (6), we get

$$D^2(z)\Theta_x - D^1(z)\Theta_y - \{D^1(z)\Theta, D^2(z)\Theta\} = 0. \tag{13}$$

The first nontrivial order of expansion of this equation at $z \to \infty$ gives exactly the heavenly equation

$$\Theta_{ty} - \Theta_{\tilde{t}x} - \Theta_{xy}^2 + \Theta_{xx}\Theta_{yy} = 0, \tag{14}$$

where $t = t_1^1$, $\tilde{t} = t_1^2$.

Identity (3) also gives a general set of addition formulae (generating equations in terms of vertex operators) for Θ [11]:

$$\begin{aligned}
&\frac{1}{z'-z}D^1(z'')(D^1(z') - D^1(z))\Theta - \frac{1}{z''-z}D^1(z')(D^1(z'') - D^1(z))\Theta \\
&\quad = D^1(z'')D^2(z)\Theta \cdot D^1(z')D^1(z)\Theta - D^1(z'')D^1(z)\Theta \cdot D^1(z')D^2(z)\Theta, \\[1ex]
&\frac{1}{z''-z}D^2(z')(D^2(z'') - D^2(z))\Theta - \frac{1}{z'-z}D^2(z'')(D^2(z') - D^2(z))\Theta \\
&\quad = D^2(z')D^2(z)\Theta \cdot D^2(z'')D^1(z)\Theta - D^2(z')D^1(z)\Theta \cdot D^2(z'')D^2(z)\Theta, \\[1ex]
&\frac{1}{z'-z}D^2(z'')(D^1(z') - D^1(z))\Theta - \frac{1}{z''-z}D^1(z')(D^2(z'') - D^2(z))\Theta \\
&\quad = D^1(z')D^1(z)\Theta \cdot D^2(z'')D^2(z)\Theta - D^1(z')D^2(z)\Theta \cdot D^2(z'')D^1(z)\Theta.
\end{aligned} \tag{15}$$

Expansion of these equations into powers of parameters z, z'', z'' generates partial differential equations for Θ of the heavenly equation hierarchy.

2.2. Generating Lax–Sato equations

We also introduce generating equations for the Lax–Sato form of the hierarchy,

$$(z' - z)D^1(z')\mathbf{S}(z) = -\{S^2(z'), \mathbf{S}(z)\}, \tag{16}$$

$$(z' - z)D^2(z')\mathbf{S}(z) = \{S^1(z'), \mathbf{S}(z)\}, \tag{17}$$

which are equivalent to the set of equations (4)–(6) (that can be checked directly or using lemma 1).

It is interesting to note that (16), (17) imply the following symmetric expressions for Poisson brackets:

$$\{S^1(z'), S^2(z)\} = 1 + (z' - z)D^2(z')D^1(z)\Theta,$$
$$\{S^1(z'), S^1(z)\} = (z - z')D^2(z')D^2(z)\Theta,$$
$$\{S^2(z'), S^2(z)\} = (z - z')D^1(z')D^1(z)\Theta.$$

3. Reductions

3.1. General N-reductions

We will first discuss the properties of general reduction, when one of the functions S^1_-, S^2_- depends on N independent functions of times (i.e., only N coefficients of expansion in z^{-1} are independent). Reductions of this type were studied a lot in dispersionless case (see, e.g., [15, 16]).

Proposition 3. *Following three statements are equivalent:*

(1)

$$S^1_-(z, \mathbf{t}^1, \mathbf{t}^2) = S^1_-(z, f_1(\mathbf{t}^1, \mathbf{t}^2), \ldots, f_N(\mathbf{t}^1, \mathbf{t}^2)), \tag{18}$$

(2)

$$\partial^2_N S^1(z, \mathbf{t}^1, \mathbf{t}^2) - \sum_{i=0}^{N-1} \phi_i(\mathbf{t}^1, \mathbf{t}^2)\partial^2_i S^1(z, \mathbf{t}^1, \mathbf{t}^2) = 0, \tag{19}$$

(3) $\frac{S^1_y}{S^1_x}$ *is a rational function with N poles,*

$$\frac{S^1_y}{S^1_x} = \sum_{i=1}^{N} \frac{u_i}{z - v_i}, \tag{20}$$

where f_i, ϕ_i, u_i, v_i *are some functions of times.*

Proof. $1 \Rightarrow 2$ is evident and it is not connected with equations of the hierarchy; it requires just some linear algebra. The absence of minus projector in (19) (in contrast with (18)) is connected with the fact that S^1 is of the form (1) and $\partial^2_i S^1_- = \partial^2_i S^1$.

$2 \Rightarrow 3$: Using equations of the hierarchy (5), one obtains

$$\frac{\partial^2_n S^1}{S^1_x} = H^2_{n\,x}\frac{S^1_y}{S^1_x} - H^2_{n\,y},$$

where $H^2_n = (z^n S^1)_+$. Substituting these expressions into relation (19) divided by S^1_x, one gets

$$\frac{S^1_y}{S^1_x} = \frac{H^2_{N\,y} - \sum_{i=0}^{N-1} \phi_i H^2_{i\,y}}{H^2_{N\,x} - \sum_{i=0}^{N-1} \phi_i H^2_{i\,x}}, \tag{21}$$

that is evidently a rational function with N poles.

$3 \Rightarrow 1$: Using equations (4), (5) and formula (20), we come to the conclusion that all ratios $\frac{\partial_n^2 S^1}{S_x^1}$, $\frac{\partial_n^1 S_-^1}{S_x^1}$ are rational functions in z with N poles in the same points that implies relation (19). Moreover, we have similar linear relations for all derivatives $\partial_k^2 S_-^1$, $N \leqslant k < \infty$, and $\partial_p^1 S_-^1$, $0 \leqslant p < \infty$, expressing these derivatives through $\partial_k^2 S_-^1$, $0 \leqslant k \leqslant N-1$. These linear equations define S_-^1 for all times through the initial data depending on the times t_k^2, $0 \leqslant k \leqslant N-1$, and the functional freedom for the general solution of the systems of linear equations of the type (19) is a function of N variables. Given a set of N functionally independent solutions $f_k(\mathbf{t}^1, \mathbf{t}^2)$, $1 \leqslant k \leqslant N$, the general solution for this system is of the form $F(f_1(\mathbf{t}^1, \mathbf{t}^2), \ldots, f_N(\mathbf{t}^1, \mathbf{t}^2))$. Thus $S_-^1(z, \mathbf{t}^1, \mathbf{t}^2)$ is of the form (18). Generically, we can take first N coefficients of expansion of $S_-^1(z, \mathbf{t}^1, \mathbf{t}^2)$ in z^{-1} as f_k, $f_k(\mathbf{t}^1, \mathbf{t}^2) = S_k^1(\mathbf{t}^1, \mathbf{t}^2)$. It is also possible to construct a set of independent solutions by the method of characteristics. □

A short comment on proposition 3: formula (18) gives a standard definition of N-reduction similar to the dispersionless case (see, e.g., [15, 16]). Equivalent formulation (19) suggests invariance of the hierarchy under the action of some vector field and it is probably useful for geometric interpretation of N-reduction. And finally, statement (3) gives analytic characterization of the reduction in terms of Lax–Sato functions. This statement also implies that all ratios $\frac{\partial_n^2 S^1}{\partial_m^1 S^1}$ are rational functions of z.

Similar statements are also known in the dispersionless case [17].

3.2. Generating equations for N-reduced hierarchy

In this subsection, we derive generating equations of N-reduced hierarchy for the functions u_i, v_i (20). We use generating Lax–Sato equations (16), (17). In the reduced case, we also have relation (20) characterizing the reduction,

$$S_y^1 = V \partial_x S^1, \qquad V = \sum_{i=1}^{N} \frac{u_i}{z - v_i}. \tag{22}$$

This linear equation for S_1 generates a compatibility condition with each of equations (16), (17) (we consider equations for $S_1(z)$). Algebraically each compatibility condition corresponds to a kind of L–A–B triad, with (22) corresponding to Lax operator. To find these compatibility conditions, we use (22) to express S_y^1 through S_x^1 in generating Lax–Sato equations, that gives linear equations of N-reduced hierarchy (plus (22)), and then consider commutation relations for *one-dimensional* (variable x) vector fields, depending on additional times. The resulting reduced hierarchy splits into two (2+1)-dimensional hierarchies.

First, to obtain linear equations of the reduced hierarchy, we use (22) to express S_y^1 through S_x^1 in generating Lax–Sato equations (16), (17),

$$(z - z')D^1(z')S^1(z) = U^1 \partial_x S^1(z), \qquad U^1 = S_x^2(z') \sum_i \frac{u_i}{z - v_i} - S_y^2(z'), \tag{23}$$

$$(z' - z)D^2(z')S^1(z) = U^2 \partial_x S^1(z), \qquad U^2 = S_x^1(z') \sum_i \frac{u_i}{z - v_i} - S_y^1(z'). \tag{24}$$

Compatibility conditions for the pairs of linear equations (22), (23) and (22), (24) are, respectively,

$$(z - z')D^1(z')V - \partial_y U^1 + V U_x^1 - V_x U^1 = 0, \tag{25}$$

$$(z' - z)D^2(z')V - \partial_y U^2 + V U_x^2 - V_x U^2 = 0. \tag{26}$$

Both equations (if we consider zero-order term at $z = \infty$) give an important relation

$$\Theta_{yy} = -\sum_i u_i \tag{27}$$

connecting S^1, S^2 with u_i using (12),

$$S^1(z') = S^1_+(z') + D^2(z')\Theta, \qquad S^2(z') = S^2_+(z') - D^1(z')\Theta. \tag{28}$$

Considering equation (25) at $z \to v_j$, one obtains a system

$$\begin{aligned} &(z' - v_j)D^1(z')u_j - u_j D^1(z')v_j = -\left(S^2_x u_j\right)_y + 2S^2_{xx}(z') \sum_{i(i\neq j)} \frac{u_i}{v_j - v_i} - \left(u_j S^2_{xy} - u_{jx} S^2_y\right) \\ &(z' - v_j)D^1(z')v_j = S^2_{xx} u_j + \left(S^2_y v_{jx} - S^2_x v_{jy}\right). \end{aligned} \tag{29}$$

Taking into account expressions (28), this is a closed (2+1)-dimensional (in a sense that it only contains the operators $D^1(z')$, ∂_x, ∂_y) system of equations generating t^1_n flows of reduced hierarchy.

Equation (25) gives a system generating flows connected with t^2_n,

$$\begin{aligned} &(z' - v_j)D^2(z')u_j - u_j D^2(z')v_j = \left(S^1_x u_j\right)_y - 2S^1_{xx}(z') \sum_{i(i\neq j)} \frac{u_i}{v_j - v_i} + \left(u_j S^1_{xy} - u_{jx} S^1_y\right) \\ &(z' - v_j)D^2(z')v_j = -S^1_{xx} u_j - \left(S^1_y v_{jx} - S^1_x v_{jy}\right). \end{aligned} \tag{30}$$

Let us consider the first systems of the reduced hierarchy. The first order of expansion of (23), (24) in z'^{-1} at infinity provides linear equations for these systems (plus (22)),

$$\begin{aligned} (\partial_t - z\partial_x)S^1 &= \left(-\Theta_{xx} \sum_i \frac{u_i}{z - v_i} + \Theta_{xy}\right) \partial_x S^1(z), \\ (\partial_{\tilde t} - z\partial_y)S^1 &= \left(-\Theta_{xy} \sum_i \frac{u_i}{z - v_i} + \Theta_{yy}\right) \partial_x S^1(z). \end{aligned}$$

The first order of expansion of (29), (30) in z'^{-1} at infinity gives the first systems of reduced hierarchy,

$$\begin{aligned} &(\partial_t - v_j\partial_x)u_j - u_j\partial_x v_j = -(\Theta_{xx}u_j)_y + 2\Theta_{xxx} \sum_{i(i\neq j)} \frac{u_i}{v_j - v_i} - (u_j\Theta_{xxy} - u_{jx}\Theta_{xy}) \\ &(\partial_t - v_j\partial_x)v_j = \Theta_{xxx}u_j + (\Theta_{xy}v_{jx} - \Theta_{xx}v_{jy}) \end{aligned} \tag{31}$$

and

$$\begin{aligned} &(\partial_{\tilde t} - v_j\partial_y)u_j - u_j\partial_y v_j = -(\Theta_{xy}u_j)_y + 2\Theta_{xxy} \sum_{i(i\neq j)} \frac{u_i}{v_j - v_i} - (u_j\Theta_{xyy} - u_{jx}\Theta_{yy}) \\ &(\partial_{\tilde t} - v_j\partial_y)v_j = \Theta_{xxy}u_j + (\Theta_{yy}v_{jx} - \Theta_{xy}v_{jy}), \end{aligned} \tag{32}$$

where Θ is defined by relation (27). A common solution to these two (2+1)-dimensional systems gives a solution Θ for the heavenly equation (14).

Now we will consider some simple special cases of the general reduction, when the function S^1 has simple analytic properties in z.

3.3. Logarithmic reduction

In this case, S^1 is of the form

$$S^1 = S^1_+ - \sum_{i=1}^{N} c_i \ln\left(1 - \frac{u_i}{z}\right).$$

Generating equations for the reduced hierarchy read

$$(z' - u_j)D^1(z')u_j = -\{D^1(z')\Theta, u_j\},$$
$$(z' - u_j)D^2(z')u_j = -\{D^2(z')\Theta, u_j\},$$
$$D^2(z')\Theta = \sum_{i=1}^{N} c_i \ln\left(1 - \frac{u_i}{z'}\right).$$

The first two (2+1)-dimensional systems of reduced hierarchy are

$$\partial_{\tilde{t}} u_k = u_k \partial_y u_k + \sum_i c_i \{u_i, u_k\}$$

and

$$\partial_t u_k = u_k \partial_x u_k - (u_k)_x \partial_x \sum_i c_i u_i - \Theta_{xx} \partial_y u_k, \qquad \Theta_y = -\sum_i c_i u_i.$$

Common solution to these systems gives a solution Θ to heavenly equation (13).

3.4. Rational reduction

We consider S^1 of the form

$$S^1 = S^1_+ + \sum_{i=1}^{N} \frac{u_i}{z - z_i}.$$

Generating equations for the reduced hierarchy read

$$(z' - z_j)D^1(z')u_j = -\{D^1(z')\Theta, u_j\},$$
$$(z' - z_j)D^2(z')u_j = -\{D^2(z')\Theta, u_j\},$$
$$D^2(z')\Theta = -\sum_{i=1}^{N} \frac{u_i}{z' - z_i}.$$

The first two (2+1)-dimensional systems of reduced hierarchy are

$$\partial_{\tilde{t}} u_k = z_k \partial_y u_k + \sum_i \{u_i, u_k\}$$

and

$$\partial_t u_k = z_k \partial_x u_k - (u_k)_x \partial_x \sum_i u_i - \Theta_{xx} \partial_y u_k,$$
$$\partial_y \Theta = -\sum_i u_i.$$

3.5. (1+1)-Dimensional reductions

If we use rational or logarithmic reduction for both S^1, S^2, we obtain (1+1)-dimensional systems of equations for coefficients directly from (6). The reduction with both S^1, S^2 rational was considered in [18].

Let us use logarithmic reduction for both S^1, S^2:

$$S^1 = S^1_+ - \sum_{i=1}^{N} c_i \ln\left(1 - \frac{u_i}{z}\right), \qquad S^2 = S^2_+ - \sum_{i=1}^{M} c_i \ln\left(1 - \frac{v_i}{z}\right).$$

Then from (6) we get a (1+1)-dimensional system of equations

$$\partial_x u_k + \sum_i c_i \frac{\{u_k, v_i\}}{u_k - v_i} = 0, \qquad \partial_y v_j - \sum_i c_i \frac{\{v_j, u_i\}}{v_j - u_i} = 0.$$

Using the expressions

$$S^1_n = \sum_{i=1}^{N} c_i \frac{u_i}{i}, \qquad S^2_n = \sum_{i=1}^{M} c_i \frac{v_i}{i},$$

we obtain the systems defining the dependence of u_k, v_j on higher times, the first two of them are

$$\begin{cases} \partial_t u_k = u_k \partial_x u_k - \sum_i c_i \{v_i, u_k\}, \\ \partial_t v_j = v_j \partial_x v_j - \sum_i c_i \{v_i, v_j\}, \end{cases} \qquad \begin{cases} \partial_{\tilde{t}} u_k = u_k \partial_y u_k + \sum_i c_i \{u_i, u_k\}, \\ \partial_{\tilde{t}} v_j = v_j \partial_y v_j + \sum_i c_i \{u_i, v_j\}. \end{cases}$$

4. Symmetry constraints

In this section, we will consider symmetries of the heavenly equation hierarchy defined through the wavefunctions of the hierarchy (solutions to linear equations of the hierarchy) and symmetry constraints connected with these symmetries. Symmetries of this type were discussed in the work [11] starting from explicit formula for the function Θ. Similar symmetry constraints are well known in the KP hierarchy case (see, e.g., [19]) as well as in the dispersionless case [14]. We will demonstrate that symmetry constraint is equivalent to rational reduction (for one of the functions S^1, S^2) of the heavenly equation hierarchy.

We introduce a set of wavefunctions $\sigma_i(\mathbf{t}_1, \mathbf{t}_2)$ depending only on the times of the hierarchy (no dependence on z),

$$(z - z_i) D^1(z)\sigma_i = -\{S^2(z), \sigma_i\}, \tag{33}$$

$$(z - z_i) D^2(z)\sigma_i = \{S^1(z), \sigma_i\}, \tag{34}$$

where z_i, $1 \leqslant i \leqslant N$, is some fixed set of points.

Proposition 4. *$\delta\Theta = \sigma_i$ is an infinitesimal symmetry for Θ (i.e., it satisfies linearized equations of the hierarchy).*

Proof. Taking vertex cross-derivatives of (33), (34), we get

$$\{S^1(z), D^1(z)\sigma_i\} + \{S^2(z), D^2(z)\sigma_i\} = 0.$$

Then, using the representation of S^1, S^2 in terms of Θ (12), we obtain

$$D^2(z)\partial_x\sigma_i - D^1(z)\partial_y\sigma_i - \{D^1(z)\sigma_i, D^2(z)\Theta\} - \{D^1(z)\Theta, D^2(z)\sigma_i\} = 0,$$

that is exactly the linearization of equation (13). In a similar manner, it is possible to prove that σ_i satisfies the linearization of a general set of addition formulae. □

Then it is possible to introduce the symmetry constraint

$$\Theta_x = \sum_{i=1}^{N} \sigma_i. \tag{35}$$

Proposition 5. *The constraint (35) is equivalent to*

$$S^2(z) = S^2_+(z) + \sum_{i=1}^{N} \frac{\sigma_i}{z - z_i}. \tag{36}$$

Proof. First, it is straightforward to demonstrate (using (12)) that constraint (35) is a necessary condition for S^2 to be of the form (36). To prove that it is sufficient, we will first prove the uniqueness of S^2 satisfying the set of linear equations associated with the heavenly equation hierarchy. □

Lemma 2. *If the function $s^2(z, \mathbf{t}^1, \mathbf{t}^2)$ satisfies linear equations*

$$(z' - z)D^1(z')s^2(z) = -\{S^2(z'), s^2(z)\} \tag{37}$$

$$(z' - z)D^2(z')s^2(z) = \{S^1(z'), s^2(z)\} \tag{38}$$

(or, equivalently, the set of linear equations associated with (4), (5)) and $(S^2)_+ = (s^2)_+$, then $s^2 = S^2$ (up to a function of z only).

Proof (lemma 2)**.** Taking (37), (38) at $z = z'$, we get

$$\{S^2(z), s^2(z)\} = 0, \qquad \{S^1(z), s^2(z)\} = 1.$$

Taking into account that $\{S^1, S^2\} = 1$, we come to the conclusion that

$$s^2(z) = S^2_+(z) + \phi,$$

where $\phi_x = \phi_y = 0$. Substituting s^2 into (37), (38) and taking into account that ϕ annulates the Poisson bracket, we obtain that

$$D^1(z')\phi(z) = D^2(z')\phi(z) = 0,$$

thus ϕ is independent of all times of the hierarchy, so it does not influence the dynamics and reflects a freedom in the definition of S^2. □

To finish the proof of proposition 5, it is enough to demonstrate that under the constraint (35) function on the rhs of (36) satisfies equations (37), (38). Substituting this function into (38), we get

$$1 + (z' - z) \sum_i \frac{1}{z - z_i} \frac{1}{z' - z_i} \{S^1(z'), \sigma_i\} = S^1_x(z') + \sum_i \frac{1}{z - z_i} \{S^1(z'), \sigma_i\}.$$

Both lhs and rhs are rational in z, the coefficients of the poles at z_i are evidently equal, and the condition at $z = \infty$ is (we use z instead of z')

$$\left(S^1_x\right)_- + \sum_1 \frac{1}{z - z_i} \{S^1, \sigma_i\} = 0.$$

Using the equations for σ_i (33), (34) to get

$$\left(S^1_x(z)\right)_- + D^2(z) \sum_i \sigma_i = 0,$$

and applying the constraint (35), we discover that the condition at $z = \infty$ is indeed satisfied, so the function $S_+^2(z) + \sum_{i=1}^N \frac{\sigma_i}{z-z_i}$ satisfies (38). In a similar manner, it is possible to prove that this function satisfies (37). Then, using lemma 2, we come to the conclusion that

$$S^2(z) = S_+^2(z) + \sum_{i=1}^{N} \frac{\sigma_i}{z - z_i}.$$

Finally, we will formulate a more general statement; the proof is completely analogous.

Proposition 6. *The constraint*

$$\partial_n^1 \Theta = \sum_{i=1}^{N} \sigma_i \tag{39}$$

is equivalent to

$$S^2(z) = S_+^2(z) + \sum_{j=1}^{n} \frac{\nu_j}{z^j} + \frac{1}{z^n} \sum_{i=1}^{N} \frac{\sigma_i}{z - z_i}, \tag{40}$$

where the functions ν_j are defined by the relations

$$\partial_{j-1}^1 \Theta = \nu_j.$$

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